BUDKERINP 94-97
December 1994

Quasiclassical Green function and Delbrück scattering in a screened Coulomb field

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Abstract

A simple integral representation is derived for the quasiclassical Green function of the Dirac equation in an arbitrary spherically-symmetric decreasing external field. The consideration is based on the use of the quasiclassical radial wave functions with the main contribution of large angular momenta taken into account. The Green function obtained is applied to the calculation of the Delbrück scattering amplitudes in a screened Coulomb field.

1 Introduction

The most convenient way to take into account the external electromagnetic field in quantum electrodynamic processes is the use of the Furry representation. So, it is necessary to know the Green function $G(\vec{r}, \vec{r}\,'|\varepsilon)$ of the Dirac equation in this field. Unfortunately, the explicit forms of the Green functions are known only for the few potentials and numerical calculations should be exploited. For many high-energy QED processes the main contribution to the amplitudes is provided by large angular momenta. Therefore, one can use the quasiclassical approximation. In the present paper, the explicit expression of the quasiclassical Green function of the Dirac equation in an arbitrary spherically-symmetric decreasing external field is found. Previously the quasiclassical Green function of the Dirac equation has been obtained in [1, 2] for the case of the Coulomb field by summing the integral representation of the exact Green function [3] over large angular momenta. As it will be shown, to obtain the quasiclassical Green function, it is not necessary to know the exact one. It is sufficient to use the quasiclassical radial wave functions at large angular momenta. This method has been applied earlier in [4] to derive Sommerfeld-Maue type wave functions [5] used at the consideration of high-energy bremsstrahlung and pair production in a screened Coulomb field. The integral representation of the Green function obtained in our paper is convenient in analytic calculations of the amplitudes of different high-energy QED processes in the external field. To confirm this statement we calculate the Delbrück scattering amplitude [6] (the elastic scattering of a photon in the external field via virtual electron-positron pairs) in a screened Coulomb field.

Delbrück scattering is one of the few nonlinear QED processes which can be tested by experiment with high accuracy (see recent review [7]). At the present time Delbrück amplitudes are studied in detail in the Coulomb field exactly in the parameter $Z\alpha$ at high photon energy $\omega \gg m$ only; m is the electron mass, $Z|e|$ is the charge of the nucleus, $Z\alpha = e^2 = 1/137$ is the fine-structure constant, e is the electron charge, $\hbar = c = 1$. The approaches used essentially depended on the momentum transfer $\Delta = |\vec{k}_2 - \vec{k}_1|$ ($\vec{k}_1$ and $\vec{k}_2$ being the momenta of the incoming and outgoing photons, respectively). At $\Delta \ll \omega$ the amplitudes have been found in [8, 9, 10] by summing in a definite approximation the Feynman diagrams with an arbitrary number of photons exchanged with a Coulomb centre, and also in [1, 2] with the help of the quasiclassical Green function in a Coulomb field. At $m \ll \Delta \sim \omega$ the amplitudes of the process have been obtained in [11, 12, 13] using the exact electron Green function in a Coulomb field [3] in the limit $m = 0$. Many authors have performed the calculations for an arbitrary photon energy ω but only in the lowest-order Born approximation with respect to the parameter $Z\alpha$ (the results obtained in this approximation are surveyed in [14]). It turned out that Coulomb corrections at $Z\alpha \sim 1$ and $\omega \gg m$ drastically change the result as compared to the Born approximation.

The effect of screening is important only in the case of small momentum transfer $\Delta \sim 1/r_c \ll m$, where r_c is the screening radius of the atom. It is this range of momentum transfer that we consider in our paper.

2 Green function

Let us consider the Green function of the Dirac equation in the external spherically-symmetric field $V(r)$:

$$G(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{1}{\gamma^0(\varepsilon - V(r)) - \vec{\gamma}\vec{p} - m + i0}\,\delta(\vec{r}-\vec{r}\,')\,, \tag{1}$$

where γ^μ are Dirac matrices, $\vec{p} = -i\vec{\nabla}$. We are interested in the calculation of the Green function at $|\varepsilon| \gg m$. Let us represent the function G in the form

$$G(\vec{r},\vec{r}\,'|\,\varepsilon) = \left[\gamma^0(\varepsilon - V(r)) - \vec{\gamma}\vec{p} + m\right] D(\vec{r},\vec{r}\,'|\,\varepsilon)\,, \tag{2}$$

where the function $D(\vec{r},\vec{r}\,'|\,\varepsilon)$ is

$$D(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{1}{(\varepsilon - V(r))^2 - \vec{p}^{\,2} - [\vec{\alpha}\vec{p}, V(r)] - m^2 + i0}\,\delta(\vec{r}-\vec{r}\,')\,. \tag{3}$$

Here $\vec{\alpha} = \gamma^0\vec{\gamma}$. As it is known (see [5]), at high energies $\varepsilon \gg m$ one can neglect $V^2(r)$ in (3) and take into account only the first term of the expansion with respect to the commutator $[\vec{\alpha}\vec{p}, V(r)]$. Making the cited expansion and using the representation

$$[\vec{\alpha}\vec{p}, V(r)] = \frac{1}{2\varepsilon}[\vec{\alpha}\vec{p}, H] \quad , \quad H = \vec{p}^{\,2} + 2\varepsilon V(r)\,, \tag{4}$$

we get the following representation for the function D :

$$D(\vec{r},\vec{r}\,'|\,\varepsilon) = \left[1 - \frac{i}{2\varepsilon}(\vec{\alpha}, \vec{\nabla} + \vec{\nabla}\,')\right] D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon)\,, \tag{5}$$

where

$$D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{1}{\kappa^2 - H + i0}\,\delta(\vec{r}-\vec{r}\,')\,, \tag{6}$$

$\kappa^2 = \varepsilon^2 - m^2$. Thus, the problem reduces to the calculation of the quasiclassical Green function $D^{(0)}$ of the Schrödinger equation with the hamiltonian H.

Let us introduce the impact parameter $\rho = |\vec{r}\times\vec{r}\,'|/|\vec{r}-\vec{r}\,'|$. In high-energy processes the characteristic distances are $|\vec{r}-\vec{r}\,'| \sim \kappa/m^2 \gg 1/m$ and $\rho \geq 1/m$. So, the corresponding angular momentum $l \sim \kappa\rho \gg 1$, and one can use the quasiclassical approximation. Besides, we consider the case $\rho \ll |\vec{r}-\vec{r}\,'|$. Hence, the angle either between $\vec{r}$ and $-\vec{r}\,'$ or between $\vec{r}$ and $\vec{r}\,'$ is small.

Consider the set of eigenfunctions of the hamiltonian H and use in (6) its completeness, replacing δ-function by the sum of products of eigenfunctions. Obviously, the main contribution to $D^{(0)}$ is provided by the functions of the continuous spectrum with large angular momentum values. The eigenfunction $\psi_{\vec{q}}(\vec{r})$, containing the plane wave with the momentum $\vec{q}$ and the outgoing spherical wave in its asymptotic form, can be represented as:

$$\psi_{\vec{q}}(\vec{r}) = \frac{1}{qr}\sum_{l=0}^{\infty} i^l \mathrm{e}^{i\delta_l}(2l+1)u_l(r)P_l(\cos\vartheta)\,. \tag{7}$$

Here $P_l(x)$ are the Legendre polynomials, ϑ is the angle between vectors $\vec{q}$ and $\vec{r}$. The set of eigenfunctions with the ingoing spherical waves in asymptotics leads to the same result for the Green function. In the quasiclassical approximation the functions $u_l(r)$ and $\delta_l = \delta(l/q)$ are equal to (see [4]) :

$$u_l(r) = \sin\left(qr - l\pi/2 + l^2/2qr + \lambda\delta(l/q) + \lambda\Phi(r)\right) , \tag{8}$$

$$\Phi(r) = \int\limits_r^\infty V(\zeta)d\zeta \quad , \quad \delta(\rho) = -\int\limits_0^\infty V\left(\sqrt{\zeta^2+\rho^2}\right)d\zeta \quad , \quad \lambda = \varepsilon/q\,.$$

Taking into account the completeness relation, we get:

$$D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon) = \int \frac{\psi_{\vec{q}}(\vec{r})\psi^*_{\vec{q}}(\vec{r}\,')}{\kappa^2 - q^2 + i0}\frac{d\vec{q}}{(2\pi)^3}\,. \tag{9}$$

Substituting (7) into (9) and taking the integral over the angles of vector $\vec{q}$ with the help of well-known relation for the Legendre polynomials

$$\int P_l(\vec{n}_1\vec{n}_2)P_{l'}(\vec{n}_1\vec{n}_3)\,d\Omega_1 = \frac{4\pi}{2l+1}P_l(\vec{n}_2\vec{n}_3)\delta_{ll'}\,,$$

where $\vec{n}_i$ are unit vectors, we obtain:

$$D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{1}{2\pi^2\,rr'}\int\limits_0^\infty \frac{dq}{\kappa^2 - q^2 + i0}\sum_{l=0}^\infty(2l+1)u_l(r)u_l(r')P_l(\vec{n}\vec{n}\,')\,, \tag{10}$$

$\vec{n} = \vec{r}/r$, $\vec{n}\,' = \vec{r}\,'/r'$. Using (8), we represent the product $u_l(r)u_l(r')$ as follows :

$$u_l(r)u_l(r') = \frac{1}{2}\cos\left[q(r-r') + l^2(r'-r)/2qrr' + \lambda(\Phi(r)-\Phi(r'))\right] \tag{11}$$
$$-\frac{1}{2}(-1)^l\cos\left[q(r+r') + l^2(r+r')/2qrr' + 2\lambda\delta(l/q) + \lambda\left(\Phi(r)+\Phi(r')\right)\right]\,.$$

If the angle θ between the vectors $\vec{n}$ and $-\vec{n}\,'$ is small, then one can replace $P_l(\vec{n}\vec{n}\,')$ by $(-1)^lJ_0(l\theta)$, where $J_\nu(x)$ is the Bessel function. After this substitution it is clear that in sum over l the main contribution comes from the second term in (11). For this term the summation with respect to l can be replaced by an integration. Let us make in (10) the exponential parametrization of the energy denominator :

$$\frac{1}{\kappa^2 - q^2 + i0} = -i\int\limits_0^\infty \exp[is(\kappa^2 - q^2)]ds$$

Then the integrals over q and s can be taken by means of the stationary phase method. After simple calculations for the case under consideration $\theta \ll 1$, one obtains:

$$D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{ie^{i\kappa(r+r')}}{4\pi\kappa rr'}\int\limits_0^\infty dll J_0(l\theta)\exp\left\{i\left[\frac{l^2(r+r')}{2\kappa rr'} + 2\lambda\delta(l/\kappa) + \lambda(\Phi(r)+\Phi(r'))\right]\right\}. \tag{12}$$

In this formula $\lambda = \varepsilon/\kappa$. In the relativistic case $\lambda = +1$ at $\varepsilon > 0$ and $\lambda = -1$ at $\varepsilon < 0$.

If the angle $\theta_1 = \pi - \theta$ between vectors $\vec{n}$ and $\vec{n}\,'$ is small then one can replace $P_l(\vec{n}\vec{n}\,')$ by $J_0(l\theta_1)$. In this case the main contribution in the sum over l comes from the first term in (11). Since it doesn't contain $\delta(l)$, it is possible to take the integral over l after the transformations similar to those performed at the derivation of (12). At $\pi - \theta \ll 1$ one has:

$$D^{(0)}(\vec{r},\vec{r}\,'|\,\varepsilon) = -\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\exp\left\{i\kappa|\vec{r}-\vec{r}\,'| + i\lambda \mathrm{sign}(r-r')(\Phi(r)-\Phi(r'))\right\} \quad . \tag{13}$$

Substituting (12) into (13) and (5), we find for the function D at $\theta \ll 1$:

$$\begin{aligned} D(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{i\mathrm{e}^{i\kappa(r+r')}}{4\pi\kappa r r'}\int\limits_0^\infty dll\left[J_0(l\theta) - i\frac{(\vec{\alpha},\vec{n}+\vec{n}\,')}{\kappa\theta}\delta'(l/\kappa)J_1(l\theta)\right] \\ \times\exp\left\{i\left[l^2(r+r')/2\kappa r r' + 2\lambda\delta(l/\kappa) + \lambda(\Phi(r)+\Phi(r'))\right]\right\} \quad . \end{aligned} \tag{14}$$

Here $\delta'(\rho) = \partial\delta(\rho)/\partial\rho$.

At $\pi - \theta \ll 1$ the function D is of the following form:

$$\begin{aligned} D(\vec{r},\vec{r}\,'|\,\varepsilon) = -\left[1 - \mathrm{sign}(r-r')(V(r)-V(r'))(\vec{\alpha},\vec{n}+\vec{n}\,')/4\kappa\right] \\ \times\exp\left\{i\kappa|\vec{r}-\vec{r}\,'| + i\lambda\mathrm{sign}(r-r')(\Phi(r)-\Phi(r'))\right\}/4\pi|\vec{r}-\vec{r}\,'| \quad . \end{aligned} \tag{15}$$

Substituting (14) and (15) into (2), we get the final result for the quasiclassical Green function of the Dirac equation in spherically-symmetric external field. One has at $\theta \ll 1$

$$\begin{aligned} G(\vec{r},\vec{r}\,'|\,\varepsilon) = \frac{i\mathrm{e}^{i\kappa(r+r')}}{4\pi\kappa r r'}\int\limits_0^\infty dll\,\exp\left\{i\left[l^2(r+r')/2\kappa r r' + 2\lambda\delta(l/\kappa) + \lambda(\Phi(r)+\Phi(r'))\right]\right\} \\ \times\left\{\left[\gamma^0\varepsilon + m - \frac{1}{2}(\vec{\gamma},\vec{n}-\vec{n}\,')(\kappa + l^2/2\kappa r r')\right]J_0(l\theta) + i\left[l^2(r-r')(\vec{\gamma},\vec{n}+\vec{n}\,')/2rr'\right.\right. \\ \left.\left. + l\delta'(l/\kappa)\gamma^0\Big(1 - (\vec{\gamma},\vec{n})(\vec{\gamma},\vec{n}\,') - (\vec{\gamma},\vec{n}+\vec{n}\,')m/\kappa\Big)\right]J_1(l\theta)/(l\theta)\right\}, \end{aligned} \tag{16}$$

and at $\pi - \theta \ll 1$

$$\begin{aligned} G(\vec{r},\vec{r}\,'|\,\varepsilon) = -\frac{1}{4\pi R}[\gamma^0\varepsilon + m - (\kappa + i/R)(\vec{\gamma},\vec{R})/R] \\ \times\exp\left\{i\kappa R + i\lambda\mathrm{sign}(r-r')(\Phi(r)-\Phi(r'))\right\} \quad , \quad \vec{R} = \vec{r}-\vec{r}\,'\,. \end{aligned} \tag{17}$$

In the Coulomb field $V(r) = -Z\alpha/r$, we have

$$2\delta(\rho) + \Phi(r) + \Phi(r') = Z\alpha\ln(4rr'/\rho^2) \quad , \quad \delta'(\rho) = -Z\alpha/\rho\,. \tag{18}$$

Using (18) and (16), we find that our result for the Green function in the Coulomb field is in agreement with that obtained in [1, 2].

3 Delbrück scattering

Let us apply the formulae obtained to the calculation of the Delbrück amplitudes in a screened Coulomb field. In the Thomas-Fermi model the screening radius $r_c \sim (m\alpha)^{-1}Z^{-1/3}$

. The characteristic impact parameter $\rho \sim 1/\Delta$. If $R \ll 1/\Delta \ll r_c$ (R is the radius of the nucleus), then the screening can be neglected and the amplitude under consideration coincides with that in the Coulomb field. If $1/\Delta \sim r_c \gg 1/m$, then it is necessary to take screening into account. For this momentum transfer the main contribution to the amplitude is provided by impact parameters ρ from $1/m$ to r_c. The corresponding angular momenta $l \sim \omega\rho \gg 1$ and the quasiclassical approximation is valid.

Let an initial photon with momentum $\vec{k}_1$ produce at the point $\vec{r}_1$ a pair of virtual particles which is transformed at the point $\vec{r}_2$ into a photon with momentum $\vec{k}_2$. Then the uncertainty relation gives $\tau \sim |\vec{r}_2 - \vec{r}_1| \sim \omega/(m^2 + \Delta^2)$ for the lifetime of the virtual pair. Therefore, at $\omega/m^2 \gg r_c$ the angles between $\vec{k}_1$, $\vec{k}_2$, $\vec{r}_2$ and $-\vec{r}_1$ are small. It is this energy range we consider further. According to the Feynman rules, in the Furry representation the Delbrück scattering amplitude reads

$$M = 2i\alpha \int d\vec{r}_1\, d\vec{r}_2 \exp[i(\vec{k}_1\vec{r}_1 - \vec{k}_2\vec{r}_2)] \int d\varepsilon\, Tr \hat{e}_2^{*} G(\vec{r}_2, \vec{r}_1|\omega - \varepsilon)\hat{e}_1 G(\vec{r}_1, \vec{r}_2| - \varepsilon) \ , \qquad (19)$$

where e_1^μ and e_2^μ are the polarization vectors of initial and final photons, respectively, $\hat{e} = e_\mu\gamma^\mu$. It is necessary to subtract, from the integrand for M in (19), the value of this integrand at zero potential. In the following such a subtraction is assumed to be made and we perform it explicitly in the final result. The main contribution to the amplitude M arises at the integration over ε from m to $\omega - m$. Thus, $\lambda = +1$ in the first Green function in (19) and $\lambda = -1$ in the second one. Using the representation (2) , it is convenient to rewrite eq. (19) in the form

$$\begin{aligned} M = i\alpha \int d\vec{r}_1 d\vec{r}_2 \exp[i(\vec{k}_1\vec{r}_1 - \vec{k}_2\vec{r}_2)] \int d\varepsilon\, Tr \Big[(2\vec{e}_2^{\,*}\vec{p}_2 - \hat{e}_2^{*}\hat{k}_2) D(\vec{r}_2, \vec{r}_1|\omega - \varepsilon)\Big] \qquad (20) \\ \times\Big[(2\vec{e}_1\vec{p}_1 + \hat{e}_1\hat{k}_1) D(\vec{r}_1, \vec{r}_2| - \varepsilon)\Big] + 2i\alpha\vec{e}_2^{\,*}\vec{e}_1 \int d\vec{r} \exp[i(\vec{k}_1 - \vec{k}_2)\vec{r}] \int d\varepsilon\, Tr D(\vec{r}, \vec{r}|\varepsilon) \ . \end{aligned}$$

Here $\vec{p}_{1,2} = -i\vec{\nabla}_{1,2}$. The last term in (20) doesn't contribute to the amplitude at high energy because it is independent of ω and depends only on momentum transfer Δ. However, the amplitude at $\omega \gg \Delta$ is proportional to ω (see, i.g., [7]). Our further transformations are as follows. We substitute (14) into (20), perform the differentiation and take the trace over γ-matrices. It is convenient to direct the axis of the spherical coordinate system along $\vec{k}_1 + \vec{k}_2$. In the small-angle approximation one has: $d\Omega_{1,2} \approx \theta_{1,2} d\theta_{1,2} d\phi_{1,2} = d\vec{\theta}_{1,2}$ with $(\vec{\theta}_{1,2}, \vec{k}_1 + \vec{k}_2) = 0$. The Bessel functions in the Green function depend on the vectors $\vec{\theta}_{1,2}$ via the combination $\theta = |\vec{\theta}_1 + \vec{\theta}_2|$ only. Let us change over to the variables $\vec{\theta} = \vec{\theta}_1 + \vec{\theta}_2$ and $\vec{\xi} = r_1\vec{\theta}_1 - r_2\vec{\theta}_2$. After that it is easy to take the integral over $d\vec{\xi}$. Further, to demonstrate the method of calculations, we consider the case of zero momentum transfer ($\vec{k}_2 = \vec{k}_1 = \vec{k}$), and present then the results of similar calculations for $\Delta \sim 1/r_c$.

3.1 Zero momentum transfer

We set $\vec{k}_1 = \vec{k}_2$, $\vec{e}_1 = \vec{e}_2$ and take the integral with respect to $d\vec{\theta}$ with the help of the relation ([15], p. 732)

$$\int\limits_0^\infty dx\, x\, e^{icx^2} J_\nu(ax) J_\nu(bx) = \frac{ie^{i\pi\nu/2}}{2c} J_\nu(ab/2c) \exp\left[\frac{-i(a^2+b^2)}{4c}\right] ,$$

and also those obtained by differentiating this expression with respect to the parameters. Let us make the substitution of the variables in the integral representation of the Green function: $l_1 = \kappa_1\rho_1$, $l_2 = \kappa_2\rho_2$ where $\kappa_1 = (\varepsilon^2 - m^2)^{1/2}$, $\kappa_2 = ((\omega-\varepsilon)^2 - m^2)^{1/2}$. After that it is convenient to pass from the variables r_1 and r_2 to s and x : $r_1 = \kappa_1\kappa_2/[m^2\omega s x]$, $r_2 = \kappa_1\kappa_2/[m^2\omega s(1-x)]$. As the result, the integral over ε becomes trivial and we get the following expression:

$$M = \frac{2i\alpha\omega m^2}{3}\int_0^1 \frac{dx}{x(1-x)}\left[1+\frac{1}{x(1-x)}\right]\int_0^\infty\int_0^\infty \rho_1\rho_2 d\rho_1 d\rho_2 \int_0^\infty \frac{ds}{s} \tag{21}$$
$$\times \exp\left\{\frac{i}{2}\left[m^2(\rho_1^2+\rho_2^2)s - 1/[sx(1-x)]\right]\right\}\sin^2(\delta(\rho_2)-\delta(\rho_1))\, J_0(m^2 s\rho_1\rho_2)\,,$$

Here we have subtracted from the integrand its value at the field equal to zero. Let us make the substitution of the variables $\rho_1 = \rho \mathrm{e}^{-\tau/2}$, $\rho_2 = \rho\mathrm{e}^{\tau/2}$ and deform the contour of the integration with respect to s so that the integral is extended from zero to $i\infty$. Then the integral over s can be taken using the relation ([15], p. 739)

$$\int_0^\infty \exp[-x(a^2+b^2)/2 \,-\, 1/2x\,]I_\nu(abx)\frac{dx}{x} = 2I_\nu(a)K_\nu(b) \quad , \quad a<b\,,$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and third kind respectively. As the result, we obtain

$$M = \frac{8i\alpha\omega m^2}{3}\int_0^1 \frac{dx}{x(1-x)}\left[1+\frac{1}{x(1-x)}\right]\int_0^\infty \rho^3 d\rho$$
$$\times\int_0^\infty d\tau\,\sin^2\Big(\delta(\rho\mathrm{e}^{\tau/2})-\delta(\rho\mathrm{e}^{-\tau/2})\Big)\, I_0(y_1)K_0(y_2)\,, \tag{22}$$

where $y_{1,2} = m\rho\mathrm{e}^{\mp\tau/2}\,[x(1-x)]^{-1/2}$.

We divide the integral over τ into two parts: from 0 to τ_0 and from τ_0 to ∞ , where $1 \gg \tau_0 \gg 1/(mr_c)$. Let us begin our calculations from the second domain. In this domain the main contribution is given by the impact parameters $\rho < r_c$, and the field can be considered as the Coulomb one. Integrating over x , ρ, and , finally, over τ, we get

$$M_2 = i\frac{28\alpha\omega}{9m^2}\int_{\tau_0}^\infty d\tau\frac{\mathrm{ch}\tau\sin^2(Z\alpha\tau)}{\mathrm{sh}^3\tau} =$$
$$-i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}[\mathrm{Re}\psi(1-iZ\alpha)+C+\ln 2\tau_0 \,-\, 3/2]\,. \tag{23}$$

Here $\psi(x) = d\ln\Gamma(x)/dx$, $C = 0.577...$ is the Euler constant.

In the first domain the difference $\delta(\rho\mathrm{e}^{\tau/2})-\delta(\rho\mathrm{e}^{-\tau/2})$ is small, and it is possible to expand the integrand with respect to this quantity. Therefore, this domain gives the contribution to the amplitude in the Born approximation only. It is convenient to divide the integral over ρ into two parts: from zero to ρ_0 and from ρ_0 to ∞, where $r_c \gg \rho_0 \gg 1/(m\tau_0)$. In

the integral from zero to ρ_0 the field can be considered again as the Coulomb one, and the integrals left can be easily taken. The corresponding contribution reads:

$$M_{11} = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln(m\tau_0\rho_0) + C - 11/21\right] . \tag{24}$$

In the integral from ρ_0 to ∞ one can use the asymptotics of the Bessel functions $I_0(x)$ and $K_0(x)$ at large x and extend the integration over τ up to infinity. The corresponding result is

$$M_{12} = i\frac{28\alpha\omega}{9m^2}\int\limits_{\rho_0}^{\infty} \rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 d\rho\,. \tag{25}$$

The sum of (23), (24) and (25) is equal to

$$M = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln(m\rho_0/2) + (Z\alpha)^{-2}\int\limits_{\rho_0}^{\infty} \rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 d\rho - \mathrm{Re}\psi(1-iZ\alpha) + \frac{41}{42}\right] . \tag{26}$$

At $\rho \ll r_c$ the integral in (25) is equal to $\ln(r_c/\rho_0)+A$, where A is some constant. Therefore, the amplitude M in (26) is independent of ρ_0 , and one can put, for instance, $\rho_0 = 2/m$. So, we have obtained the final result for the forward Delbrück scattering amplitude for an arbitrary screened potential. The explicit value of the constant depends on the form of the potential.

Let us consider the case of the Molière potential [16], which approximates the potential in the Thomas-Fermi model:

$$V(r) = -\frac{Z\alpha}{r}\sum_{i=1}^{3}\alpha_i \mathrm{e}^{-\beta_i r}\,, \tag{27}$$

where $\alpha_1 = 0.1$, $\alpha_2 = 0.55$, $\alpha_3 = 0.35$, $\beta_i = mZ^{1/3}b_i/121$, $b_1 = 6$, $b_2 = 1.2$, $b_3 = 0.3$. The corresponding scattering phase is equal to

$$\delta(\rho) = Z\alpha\sum_{i=1}^{3}\alpha_i K_0(\beta_i\rho)\,. \tag{28}$$

Substituting this expression into (26), we get the final result for the forward Delbrück scattering amplitude in the Molière potential:

$$M = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln(183Z^{-1/3}) - C - \mathrm{Re}\psi(1-iZ\alpha) - \frac{1}{42}\right] . \tag{29}$$

As known, the imaginary part of the forward scattering amplitude of the photon is connected with the total cross section σ of electron-positron pair production by the relation $\sigma = \mathrm{Im}\,M/\omega$. Due to this relation, our formula (29) is in agreement with the result of [17] for the total cross section of pair production in a screened potential. Note that the real part of the amplitude (29) in the screened Coulomb potential is equal to zero in contrast to the case of pure Coulomb potential [9, 2].

3.2 Non-zero momentum transfer

At non-zero momentum transfer it is convenient to carry out the calculation in terms of helicity amplitudes. One can choose the polarization vectors in the form

$$\vec{e}^{\,\pm}_{1,2} = ([\vec{\lambda} \times \vec{\nu}_{1,2}] \pm i\vec{\lambda})/\sqrt{2}\,, \quad \vec{\lambda} = [\vec{\nu}_1 \times \vec{\nu}_2]/|[\vec{\nu}_1 \times \vec{\nu}_2]|\,, \tag{30}$$

where $\vec{\nu}_{1,2} = \vec{k}_{1,2}/\omega$. There exist two independent amplitudes: $M^{++} = M^{--}$ and $M^{+-} = M^{-+}$. In terms of linear polarization, by virtue of parity conservation, the amplitude differs from zero only when the polarization vectors of the initial and final photon both lie in the scattering plane ($M^{\parallel}$) or are perpendicular to it ($M^{\perp}$). These types of amplitudes are related via

$$M^{\parallel} = M^{++} + M^{+-}\,,\; M^{\perp} = M^{++} - M^{+-}\,.$$

At $\Delta = 0$ the amplitude M^{+-} vanishes by virtue of the conservation of the angular momentum projection along the direction of motion of the initial photon, and the amplitude M^{++} coincides with (29) . Similar to the case of zero momentum transfer, we divide the integral over τ into two parts: from 0 to τ_0 and from τ_0 to ∞ , where $1 \gg \tau_0 \gg 1/(mr_c)$. The angle θ_0 between $\vec{k}_1$ and $\vec{k}_2$ is $\theta_0 = \Delta/\omega \ll m/\omega$. In the domain from τ_0 to ∞ the field concides with the Coulomb one and the angle θ_0 can be neglected. The contribution of this domain to the amplitude M^{++} coincides with M_2 (29) , and the contribution to the amplitude M^{+-} is equal to zero. In the domain from zero to τ_0 we split the integral over ρ into two parts again: from zero to ρ_0 and from ρ_0 to ∞, where $r_c \gg \rho_0 \gg 1/(m\tau_0)$. In the integral from zero to ρ_0 the field can be treated as a Coulomb one and the angle θ_0 can be neglected again. The corresponding contribution to M^{++} coincides with M_{11} (24) , and the contribution to M^{+-} is equal to zero. The effect of screening is essential in the last domain from ρ_0 to ∞ only. In this domain the main contribution to the integral over angles is given by $\theta \sim \rho/r \sim \rho m^2/\omega \gg m/\omega \gg \theta_0$. The argument of the Bessel functions in the expression for the Green function is $l\theta \sim \omega\rho\theta \sim (m\rho)^2 \gg 1$, and one can use the asymptotic expansion of the Bessel functions. It is necessary to keep two terms of the expansion due to the compensation. After that the integrals over θ and the other variables can be easily taken, and we get the contribution of the domain under discussion to the amplitude M^{++} :

$$M^{++}_{12} = i\frac{28\alpha\omega}{9m^2}\int\limits_{\rho_0}^{\infty} \rho \left(\frac{\partial\delta}{\partial\rho}\right)^2 J_0(\rho\Delta)\, d\rho\,. \tag{31}$$

The sum of (31), (23) and (24) is:

$$M^{++} = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[(Z\alpha)^{-2}\int\limits_{2/m}^{\infty} \rho \left(\frac{\partial\delta}{\partial\rho}\right)^2 J_0(\rho\Delta)d\rho \,-\mathrm{Re}\psi(1-iZ\alpha) + \frac{41}{42}\right]. \tag{32}$$

Substitute (28) into (32) and take the integral over ρ by means of formula (6.578(10)) [15]. Then, the final result for the amplitude M^{++} in the case of Molière potential reads:

$$M^{++} = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\Big\{-\mathrm{Re}\psi(1-iZ\alpha) - C + \frac{41}{42} \tag{33}$$
$$-\frac{1}{2}\sum_{i,j}\alpha_i\alpha_j\left[\ln(\beta_i\beta_j/m^2) + \frac{u}{\sqrt{u^2-1}}\ln(u+\sqrt{u^2-1})\right]\Big\},$$

where $u = (\Delta^2 + \beta_i^2 + \beta_j^2)/2\beta_i\beta_j$. At $\Delta \ll 1/r_c$ the formula (33) coincides with (29). At $m \gg \Delta \gg 1/r_c$ the formula (33) turns into

$$M^{++} = i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left\{\ln\frac{m}{\Delta} - \mathrm{Re}\psi(1 - iZ\alpha) - C + \frac{41}{42}\right\}, \tag{34}$$

in agreement with the result of [8, 9, 2]. Similarly, for the amplitude M^{+-}, we obtain:

$$M^{+-} = i\frac{4\alpha\omega}{9m^2}\int\limits_0^\infty \rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 J_2(\rho\Delta)\,d\rho\,. \tag{35}$$

Here the lower limit of the integral is replaced by zero since the domain from zero to ρ_0 doesn't contribute. For the Molière potential one has

$$\begin{aligned} M^{+-} = i\frac{2\alpha\omega(Z\alpha)^2}{9m^2}\Big\{1 + \frac{1}{\Delta^2}\sum_{i,j}\alpha_i\alpha_j\Big[(\beta_i^2 - \beta_j^2)\ln\frac{\beta_i}{\beta_j} \\ - \frac{u(\beta_i^2 + \beta_j^2) - 2\beta_i\beta_j}{\sqrt{u^2-1}}\ln(u + \sqrt{u^2-1})\Big]\Big\}. \end{aligned} \tag{36}$$

If $\Delta \to 0$ then M^{+-} (36) tends to zero. At $m \gg \Delta \gg 1/r_c$ the formula (36) turns into

$$M^{+-} = i\frac{2\alpha\omega(Z\alpha)^2}{9m^2}\quad, \tag{37}$$

which is in accordance with the result of [8, 9, 2].

Thus, we have demonstrated on the example of Delbrück scattering in a screened Coulomb potential that the quasiclassical Green function obtained in the present paper can be used effectively at the consideration of high-energy QED processes in an arbitrary spherically-symmetric decreasing external field.

We are grateful to V.M.Katkov and V.M.Strakhovenko for useful discussions.

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